The Complete Information First–Price Auction or the Importance of Being Indivisible[*]

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Abstract

Despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the first-price sealed-bid auction mechanism under complete information. This paper aims to remedy that omission. We show that the existence of pure strategy undominated Nash equilibria requires that the bidding space is not "too divisible" (that is, a continuum). In fact, when bids must form part of a finite grid there always exists a "high price equilibrium". However, there might also be "low price equilibria" and when the bidding space is very restrictive the revenue obtained in these "low price equilibria" might be very low. We discuss the properties of the equilibria and an application of auction theoretical thinking in which "low price equilibria" may be relevant.

Keywords: First-price auctions, undominated Nash equilibria.

JEL Classification Numbers:

C72 (Noncooperative Games),

D44 (Auctions).

"We restrict attention to the case of complete information and try to understand this model more fully and provide a number of results in this area in an elementary manner."

Krishna and Tranaes (2002)

1. Introduction

Auction-theoretic ways of thinking have been successfully applied to the analysis of broader economic questions, including price-setting behavior in markets, litigation systems or financial crashes (see Klemperer (2003)). Depending on the desired application, the appropriate model uses the benchmark of complete information or an incomplete information setting. For instance, Moldovanu and Sela (2003) use the first-price auction mechanism to model patent licensing. In their view the benchmark of complete information is appropriate for mature industries, like the steel industry. For this industry they report that competitors "know each other well, and engineers often visit competitors' plants". However, they argue that emerging or very dynamic and secretive industries, like the petrochemical industry, are better captured by an incomplete information model.

Despite the popularity of auction theoretical thinking, it appears that no one has presented an elementary equilibrium analysis of the first-price sealed-bid auction mechanism under complete information. This paper aims to remedy that omission.[1] As a starting point we show that in the standard model with a continuous bidding space there exists no undominated Nash equilibrium. This is due to the discontinuity of the payoff functions when two bidders tie at the highest bid and parallels a well-known result in the related game of Bertrand competition.[2]

There are two prominent ways to restore existence of equilibrium. The first is to look for mixed-strategy equilibria and the second–this is the approach we take in the present paper–is to assume that bids must be chosen from a finite grid.[3] Another reason why a careful analysis under the assumption of a finite grid is important is the fact that this model is often viewed as a better description of reality.[4] Notice that in experimental settings there is also a smallest monetary unit.

[1] Although complete information settings are often considered to be a useful starting point of the analysis before moving to incomplete information models (see e.g. Baye et al. (1993) and (1996), Benoit and Krishna (2001), Bernheim and Whinston (1986) or Krishna and Tranaes (2002)) or to be a useful benchmark case (see e.g. Anton and Yao (1989), Grimm et al. (2003), Elmaghraby and Oren (1999), or Moldovanu and Sela (2003)), we are not aware of any work on the first-price auction with complete information that contains the elementary results that we derive in the present paper.

[2] A first-price auction and a Bertrand oligopoly market in which firms produce a homogenous good at constant marginal costs are in general not equivalent, as in the latter the winning firm's demand decreases in the price. This corresponds to an auction in which the winning bidder's valuation increases in her bid.

[3] See Blume (2003) for a recent contribution to the analysis of mixed-strategy equilibria in Bertrand competition and Vives (2001) for a discussion of a rationale for mixed-strategies in this model. Other auction models using the assumption of a finite grid are Shubik (1971), O'Neill (1986), Chwe (1989) or Rapoport and Amaldoss (2004).

[4] Simon and Zame (1990, p. 863) state this view as follows. "Games with infinitely many strategies are sometimes viewed as proxies for games with a large finite number of strategies. From this point of view it is the equilibria ... of the finite games which are of real interest; equilibria of the infinite games are merely convenient approximations." Rapoport and Amaldoss (2004, p. 587) write "the assumption of a discrete strategy space is appropriate as firms typically consider their expenditures in discrete (e.g., thousands or millions of dollars) rather than continuous units. Indeed, continuous strategy spaces are mostly introduced to achieve tractability, not to provide a more adequate description of reality".

The first contribution of the present paper is to offer an elementary analysis of pure strategy undominated Nash equilibria in the first-price sealed-bid auction mechanism under complete information. Our approach uses fairly general tie-breaking rules and (possibly irregular) finite grids on bidding spaces. We find that the existence of a "high price equilibrium" in which

§ the object will be awarded to the bidder with the highest valuation at a price close to the second highest valuation and

§ the outcome is efficient

requires precise condition on the bidding space to be fulfilled. Namely, the bidding space should not be too divisible (a continuum) on one hand and not be too indivisible on the other.

Our second contribution is to show that there might also be "low price equilibria" and that these do not automatically go away as the smallest monetary unit diminishes. Moreover, when the bidding space is very restrictive the revenue obtained in these "low price equilibria" might be very low.

Clearly the question of whether a restrictive bidding space is the appropriate assumption to make depends on the application one has in mind.[5] Consider several jurisdictions

[5] For instance, in Bertrand competition the smallest monetary unit is arguably small. So "low price equilibria" should be less important. Moreover, as demand decreases with the price charged (in our terminology, the winning bidder's valuation increases in her bid) there is an additional incentive to undercut (that is, outbid) the rivals and "low price equilibria" should be less stable.

competing for the location of a new factory (see Menezes (2003) and Taylor (1992)). Different types of competition can be modeled by different auction formats. Investments in infrastructure already incurred can be captured by an all-pay auction, while a first-price auction might represent e.g. commitments to spend on infrastructure. In a model of incomplete information Menezes (2003) shows that the expected total amount paid to the firm under a large family of auctions is the same.

However, if jurisdictions are neighbors or relatively similar in the characteristics determining the value from the factory, the complete information setting is a reasonable benchmark case.[6] Moreover, in such a situation bids are different bundles containing commitments to spend on infrastructure, tax incentives, etc. This implies that although these bundles can be evaluated in US dollars, the bidding space may become irregular. Either a jurisdiction commits to constructing a regional airport, an entrance to a motorway or to expand the harbor or not. It cannot provide just a fraction of an airport. Our analysis implies that if the value of the infrastructure commitment is relatively high in comparison to the value of the firm to the jurisdictions, then there may be a "low price equilibrium" without commitment to build this infrastructure. Therefore, the expected total amount paid is not always the same but depends on the details of the competition.

[6] Alternatively one may think that the consultant's reports that are the private information in Menezes (2003) become publicly known.

2. The Model

2.1. The Agents

Let us consider an (indivisible) object owned by an individual whom we will call the seller (and index by 0). The object is likely to be purchased by a set $\mathcal{B} = \{B_1, \ldots, B_i, \ldots, B_n\}$ of agents, to be called the buyers. Each agent has a valuation v_i for the object. For the sake of simplicity, we will assume that there are at least two buyers, i.e. $n \geq 2$, and agents' valuations are increasingly ordered, i.e.

$$v_i \leq v_j \text{ for all } 0 \leq i \leq j.$$

The seller decides to auction the object, using a first-price auction (to be specified later on). Therefore, if the object is sold at price p, each agent's (indirect) utility is

(*i*) $r_0 = p - v_0$ for the seller,

(*ii*) $r_i = v_i - p$ for the buyer who gets the object, and zero for all the other (potential) buyers.

The informational setting that we will consider is that agents' valuations $v = (v_0, v_1, \ldots, v_i, \ldots, v_n)$ are commonly known by all the buyers, and that this is public information. As in Bernheim and Whinston (1986) or Anton and Yao (1989) we also assume that the seller only has information about her own valuation of the object.

2.2. The Bidding Space

Let A be the (fixed) set of prices that each buyer can propose. Our analysis will distinguish two scenarios:

We say that the bidding space is continuous when any non-negative amount of money is a feasible bid. Formally, we suppose that $A \equiv \mathbb{R}_+$.

We say that the bidding space is discrete when A has a finite grid satisfying the following property:

Assumption 1: There exist $\delta > 0$ such that, for all $a, a' \in A$, $a \neq a'$, $|a - a'| \geq \delta$.

Let us observe that, if bids are expressed in US dollars, the set of possible bids satisfies the Assumption 1: two different bids must differ in, at least, one cent! We may then order the set A, and denote it by $A = \{a_0, a_1, ..., a_k, ...\}$. Note that we do not require all successive elements a_k and a_{k+1} of the bidding space to be separated by regular intervals. If this is the case we will say that the bidding space has a constant grid of (at least) size δ.

2.3. The First-Price Auction Mechanism

We formalize now the first-price auction mechanism analyzed in the present paper. Loosely speaking, the object is assigned to the buyer with highest bid, and she pays her bid. However, when two or more buyers propose the same bid there is a function π establishing a probabilistic allocation rule in order to assign the object. This fixed monotonic (probabilistic) measure function

$$\pi : 2^{\mathcal{B}} \to \mathbb{R}^n$$

satisfies:

(a) for all $S \subseteq \mathcal{B}$, $\sum_{i=1}^{n} \pi_i(S) = 1$,

(b) for all $S \subseteq \mathcal{B}$ and $i \in \mathcal{B} \backslash S$, $\pi_i(S) = 0$,

(c) for all $S \subseteq \mathcal{B}$ and $i \in S$, $\pi_i(S) > 0$; and

(d) for all $S \subseteq S' \subseteq \mathcal{B}$, and $i \in S$, $\pi_i(S) \geq \pi_i(S')$.

For A and π given, the procedure describing how buyers compete for the object can be formalized as a two-stage game. At the first stage, each buyer simultaneously sets the price $p_i \in A$ $(i = 1, \ldots, n)$ that she will pay for the object if it is assigned to her. This defines a vector $p = (p_1, \ldots, p_n)$. Then, at the second stage the seller establishes her reservation price, p_0. Given these actions, the first-price auction mechanism proceeds as follows:

(1) If $p_i < p_0$ for all $i = 1, \ldots, n$, the object is unassigned, i.e. the seller keeps it.

(2) Otherwise, denote by $S(p) = \{B_i \in \mathcal{B} : p_i \geq p_j \text{ for all } B_j \in \mathcal{B}\}$ the set of buyers proposing the highest bid. Then the object is assigned with probability $\pi_i(S(p))$ to buyer B_i who pays p_i with this probability.

2.4. The Equilibrium Concept

Let us observe that the only role of stage two is to guarantee that the seller avoids losses (or negative utility). Note that the equilibrium concept that is most commonly analyzed for this type of games is subgame perfect equilibrium (SPE). Let us observe that the best strategy for the seller is to behave truthfully. More precisely, at any SPE, the object will be sold whenever some buyer sets a price higher than v_0. In fact this is a dominant strategy for the seller. Therefore, we will concentrate on the analysis of buyers' decisions at the first stage.

We will concentrate on the analysis of undominated Nash equilibria (in pure strategies) in the bidding subgame. Note that for each buyer i, a strategy $\hat{p}_i$ is undominated if, and only if, $0 \leq \hat{p}_i < v_i$.[7]

3. Analysis of the First-Price Auction Mechanism

3.1. First-Price Auctions when the Bidding Space is a Continuum

Let us consider that $A \equiv \mathbb{R}_+$. In this case we have the following result:

Theorem 3.1. *Let $A \equiv \mathbb{R}_+$. There is a Nash equilibrium for the first-price auction if, and only if, $v_n = v_{n-1}$.*

[7] The main reason for our focus on undominated Nash equilibria is to rule out 'unreasonable' equilibria. Think of the discrete setting and suppose that only integer bids are feasible. Assume there are two buyers valuing the object in $v_1 = 1$ and $v_2 = 10$, while the seller's reserve price is zero. For instance, $b_1^* = 7$ and $b_2^* = 8$ constitute a Nash equilibrium (as long as π is not too biased towards B_2).

Proof. First, it is straightforward to see that, when $v_n = v_{n-1}$ the strategy profile p^* in which each buyer B_i bids $p_i^* = v_i$ is a Nash equilibrium for this game.

Now, let us assume that $v_n > v_{n-1}$. We will proceed by contradiction. Let us suppose that there is a Nash equilibrium. Let p^* be such an equilibrium. Let us observe that p^* must satisfy that for each bidder B_i, $p_i^* \leq v_n$. Denote (one of) the highest bidders other than B_n by B_k, where B_k is such that

$$p_k^* = \max_{j \neq n} p_j^*.$$

Let us consider the following cases:

(*a*) $\pi_n(S(p^*)) = 0$. This only happens if $p_n^* < p_k^*$. Note that, as $v_k < v_n$, B_k only obtains a non-negative utility if $p_k^* < v_n$. Notice that bidder B_n is not employing a best response. This is because by setting $\tilde{p}_n = p_k^*$ we have that $\pi_n\left(S\left(\tilde{p}_n, p_{-n}^*\right)\right) > 0$ and thus the (expected) utility of agent B_n is

$$\pi_n\left(S\left(\tilde{p}_n, p_{-n}^*\right)\right)[v_n - \tilde{p}_n] > 0 = \pi_n\left(S\left(p_n^*, p_{-n}^*\right)\right)[v_n - p_n^*].$$

(*b*) $0 < \pi_n(S(p^*)) < 1$. This only happens if $p_n^* = p_k^*$ and thus, agent B_n's utility is

$\pi_n(S(p^*))[v_n - p_n^*] > 0$.[8] Let us consider

$$\tilde{p}_n = p_n^* + \frac{1 - \pi_n(S(p^*))}{2}(v_n - p_n^*).$$

Since $\tilde{p}_n > p_n^*$, we have that $\pi_n\left(S\left(\tilde{p}_n, p_{-n}^*\right)\right) = 1$, and thus agent B_n's utility is

$$\begin{aligned} v_n - \tilde{p}_n &= v_n - p_n^* - \frac{1 - \pi_n(S(p^*))}{2}(v_n - p_n^*) = \\ &= \frac{1 + \pi_n(S(p^*))}{2}(v_n - p_n^*) > \pi_n(S(p^*))(v_n - p_n^*); \end{aligned}$$

which shows that p_n^* is not an optimal decision for buyer B_n.

(*c*) $\pi_n(S(p^*)) = 1$. Let us observe that, in such a case, it holds that $p_n^* > p_k^*$. Therefore, agent B_n can also obtain the object, with probability 1, by bidding

$$\tilde{p}_n = \frac{p_n^* - p_k^*}{2}$$

since $\tilde{p}_n < p_n^*$, we have that p_n^* is not an optimal strategy given others' bids. ■

Note that a Nash equilibrium only exists for very special profiles of bidders' valuations. Moreover, in every such equilibrium at least one B_j with $v_j = v_n$ bids $p_j^* = v_j$ and the seller extracts the full surplus $v_n - v_0$. Notice also that bidding $p_i^* = v_i$ is a

[8] Again, the strict inequality comes from the fact that agent B_k cannot obtain negative (expected) utility at equilibrium.

dominated strategy which implies the following Corollary.

Corollary 3.2. *Let $A \equiv \mathbb{R}_+$. There is no undominated Nash equilibrium for the first-price auction.*

3.2. First-Price Auctions when the Bidding Space is Discrete

Consider the case in which the bidding space satisfies Assumption 1. We first need some additional notation. Denote by $w_i \in A$ agent B_i's largest (undominated) bid strictly smaller than v_i.

Consider $a_k \in A$. Denote the distance to the next element of the bidding space by $\delta_{a_k} = a_{k+1} - a_k$. For simplicity we also denote $\delta_{w_{n-1}} = \Gamma$. It will turn out that if a strategy profile is an equilibrium, then it belongs to the following class of strategy profiles.

Definition 3.3. *Given $a \in A$, $a \leq \min\{w_{n-1} + \Gamma, w_n\}$, we denote by $\mathcal{P}(a)$ the set of strategy profiles $\hat{p}$ is such that:*

1. *Buyer B_n chooses $\hat{p}_n = a$.*

2. *There exists $B_j \in \mathcal{B} \setminus \{B_n\}$ such that $w_j = w_{n-1}$ bidding $\hat{p}_j = \min\{a, w_{n-1}\}$.*

3. *All other bidders $B_i \in \mathcal{B} \setminus \{B_j, B_n\}$ choose $\hat{p}_i \leq \min\{w_i, a\}$.*

Notice that $a \in A$ just indicates the winning bid.[9] Given a strategy profile $\hat{p} \in \mathcal{P}(a)$, we indicate the buyers bidding at least $b \in A$ by $W(b) = \{B_i \in \mathcal{B} \text{ s.t. } \hat{p}_i \geq b\}$. To simplify notation we will omit a and b using δ and W instead, whenever this notation is clear from the context.

Given a winning bid a and a strategy profile $\hat{p} \in \mathcal{P}(a)$, it might pay to raise or lower an individual bid. Define the two threshold values

$$\begin{aligned} \alpha &= [1 - \pi_n (W(w_{n-1}))] [v_n - w_{n-1}] \qquad \text{and} \\ \beta &= \max \quad [1 - \pi_j (W(a))] [v_j - a] \\ & \quad\ \ s.t \qquad B_j \in W(a). \end{aligned}$$

Notice that the definition of β might not be determined by B_n when the probabilistic measure function π is strongly biased in favor of this buyer.[10]

We are now in a position to characterize undominated Nash equilibria when the bidding space has a finite grid. There are three cases to be distinguished. Case (1) and case (2.2) formalize the conventional wisdom that the strongest bidder just outbids the others or ties with an equally strong bidder at their common valuation. However, case (2.1) shows that even when valuations are different it might not pay to outbid others because the required increase of the bid may be too large. Case (3) establishes that this

[9] We implicitly assume in what follows that $v_0 \leq a$.

[10] The exact threshold for B_n not to determine β is that there exists $B_i \in W \backslash B_n$ such that $\pi_n > 1 - (1 - \pi_i)(v_i - a)/(v_n - a)$.

intuition might even apply to much lower bids.

Theorem 3.4. *Let Assumption 1 hold. A profile of strategies p^* is an undominated Nash equilibrium for the first-price auction if, and only if, $p^* \in \mathcal{P}(a)$ for some $a \in A$; and one of the following is true:*

(1) (High price equilibrium, unique winner) $a = w_{n-1} + \Gamma$, $w_{n-1} < w_n$ and $\Gamma \leq \alpha$.

(2) (High price equilibrium, tie) $a = w_{n-1}$ and either

(2.1) $w_{n-1} < w_n$ and $\Gamma \geq \alpha$ or

(2.2) $w_{n-1} = w_n$.

(3) (Low price equilibrium, tie) $a < w_{n-1}$ and $\delta_a \geq \beta$.

Proof. (I) We show first that p^* is an undominated Nash equilibrium for the first-price auction whenever (1), (2) or (3) are true. Note that, since $p_i^* \leq w_i$ for all $B_i \in \mathcal{B}$, no buyer employs a dominated strategy. We show now that p^* is a Nash equilibrium. Let us observe that the expected utility of buyers in $\mathcal{B} \setminus W$ is zero. Moreover, given agents' bids, no buyer in $\mathcal{B} \setminus W$ can obtain a positive (expected) utility.

Suppose (1) holds. The fact that $p^* \in \mathcal{P}(a)$ implies that B_n wins, so

$$U_n(p^*) = v_n - a \geq 0, \text{ with } a = w_{n-1} + \Gamma.$$

Assume B_n changes p_n^* to $\tilde{p}_n$. Given that she cannot gain from raising her bid, suppose $\tilde{p}_n \leq w_{n-1}$. Notice that there exists $B_j \neq B_n$ bidding $\hat{p}_j(a) = w_{n-1}$. We have

$$\begin{aligned} U_n\left(\tilde{p}_n, p_{-n}^*\right) &\leq \pi_n\left(W(w_{n-1})\right)\left[v_n - w_{n-1}\right] = v_n - w_{n-1} - \alpha \leq \\ &\leq v_n - w_{n-1} - \Gamma = v_n - a = U_n\left(p^*\right). \end{aligned}$$

And, thus, p_n^* is the best decision for agent B_n, given the others' bids.

Suppose (2) or (3) holds. For buyer $B_j \in W$, we have that her expected utility is

$$U_j\left(p^*\right) = \pi_j\left(W\right)\left[v_j - p_j^*\right] = \pi_j\left(W\right)\left[v_j - a\right] > 0.$$

Assume B_j changes her strategy, by setting $\tilde{p}_j$. If she lowers her bid, her expected utility will be zero, since W is not a singleton. Thus, suppose $\tilde{p}_j > a$, and note that B_j will get the object with probability one. Notice that in case (2.2) $U_j\left(\tilde{p}_j, p_{-j}^*\right) < 0$. Consider case (3). Observe that

$$\begin{aligned} U_j\left(\tilde{p}_j, p_{-j}^*\right) &= v_j - \tilde{p}_j \leq v_j - a - \delta_a \leq v_j - a - \beta \leq \\ &\leq v_j - a - \left[1 - \pi_j\left(W\right)\right]\left[v_j - a\right] = U_j\left(p^*\right). \end{aligned}$$

Again, p_j^* is the best decision for agent B_j, given the others' bids. The argument for case (2.1) is similar replacing B_j, δ_a and β by B_n, Γ and α respectively.

(II) We show now the converse. Suppose there is a Nash equilibrium p^* in which all agents employ undominated strategies. Denote the highest bid by

$$\begin{aligned} a' \quad = \quad & \max \quad p_i^* \\ & s.t \qquad B_j \in \mathcal{B} \end{aligned}$$

Notice that if $a' > \min\{w_{n-1} + \Gamma, w_n\}$, then a' is either dominated or $p_n^* = a'$. In the latter case B_n can improve by lowering her bid. Hence, suppose $a' \leq \min\{w_{n-1}+\Gamma, w_n\}$. Notice that $p_n^* = a'$ must hold because otherwise B_n can improve by making this bid. Suppose $a' = w_{n-1} + \Gamma$ and that there does not exist $B_j \in \mathcal{B} \setminus \{B_n\}$ such that $w_j = w_{n-1}$ bidding $p_j^* = w_{n-1}$. Given that for bidders with lower valuations $p_i^* = w_{n-1}$ is dominated, B_n could improve by lowering her bid. Assume $a' \leq w_{n-1}$ and that there exists $B_j \in \mathcal{B} \setminus \{B_n\}$ such that $w_j \geq a'$ bidding $p_j^* < a'$. In this case B_j can improve by changing her bid to $\tilde{p}_j = a'$ because

$$U_j\left(p^*\right) = 0 < \pi_j\left(W(a') \cup B_j\right)\left[v_j - a'\right] = U_j\left(\tilde{p}_j, p_{-j}^*\right)$$

This proves that $p^* \in \mathcal{P}\left(a'\right)$. From part (I) it is clear p^* cannot be a undominated Nash equilibrium when the conditions in case (1), (2) or (3) are not fulfilled. ■

Observe that case (1) and case (2) of Theorem 3.4 imply the following.

Corollary 3.5. *Under Assumption 1 there exists an undominated Nash equilibrium* *in the first-price auction. In this high price equilibrium the winning bid* a *fulfills* $a \in \{w_{n-1}, w_{n-1} + \Gamma\}$.

However, in addition to this equilibrium there might be further low price equilibria as specified in case (3) of Theorem 3.4. The intuition for the existence of the low price equilibria is the following. A bidder can prevent a tie by outbidding the competitors by the minimal increase. However, when the grid is restrictive the required increase is large and does not pay. So a natural question to ask is, How small must a smallest monetary unit be in order to make sure that low price equilibria do not exist? Assume that the tie breaking rule assigns the object with equal probability and that there are two bidders.[11]

Corollary 3.6. *Assume that there are two bidders who get the object with equal probability in case of a tie and that the bidding space has a constant grid of size* δ. *For any* $\delta > 0$ *the following is true:*

(1) If $w_{n-1} < w_n$, *the high price equilibrium is unique.*

(2) If $w_{n-1} = w_n$, *in addition to the high price equilibrium, there exists a low price equilibrium with strategy profile*[12] $\hat{p}(a)$ *where* $a = w_{n-1} - \delta$.

[11] Notice that because of the monotonicity of the tie breaking rule further bidders increase the incentives to deviate from a low price equilibrium. For completeness we mention that case (1) of the next Corollary assumes that $\Gamma \neq \alpha$.

[12] Note that, for the two-bidder case, for any $a \in A$ such that $a \leq \min\{w_{n-1} + \Gamma, w_n\}$, $\mathcal{P}(a)$ is a singleton. Therefore, as we do throughout this corollary and its proof, we can denote by $\hat{p}(a)$ such an element.

Proof. Suppose $w_{n-1} < w_n$. The fact that $v_n > w_n \geq w_{n-1} + \delta$ implies that $\delta < v_n - w_{n-1}$. We show first that the profile $\hat{p}(a)$ with $a = w_{n-1} - \delta$ is not an equilibrium. For this $\delta < \beta$ must hold. Since $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta)$, $\delta < \frac{1}{2}(v_n - w_{n-1} + \delta)$ must hold. Simplifying yields $\delta < v_n - w_{n-1}$, which we already have shown to be true. Notice that no profile $\hat{p}(a')$ with $a' < w_{n-1} - \delta$ can be an equilibrium because $\beta(a) = \frac{1}{2}(v_n - a) < \frac{1}{2}(v_n - a') = \beta(a')$.

Suppose $w_{n-1} = w_n$. Notice that $v_n - w_n \leq \delta$. This implies that $\beta = \frac{1}{2}(v_n - w_{n-1} + \delta) \leq \delta$. ■

This result implies, on one hand, that whenever $v_{n-1} = v_n$, there are at least two equilibria. Note also that for the second statement to hold it is not needed that the grid is constant. It is sufficient that the discrete jump immediately before w_{n-1} is larger or equal than the one after w_{n-1}. On the other hand, when valuations are different decreasing the size of the smallest monetary unit assures that up from a certain point $w_{n-1} > w_n$ holds and the high price equilibrium is unique. We give now an example in which increasing the restrictiveness of the bidding space creates multiple equilibria.

Example 3.7. *There are two bidders with valuations $v_1 = 90$ and $v_2 = 100$. The reservation price of the seller is zero. In the case that both bidders submit the same bid, they obtain the object with equal probability. Suppose first that the bidding space coincide with the set of uneven integers. In this case Theorem 3.4(1) and Corollary 3.6(1) imply that $p^* = (89, 91)$ is the unique undominated Nash equilibrium. However,* *if bidding space is* $A = \{1, 51, 76, 89, 91, 99, 106, ...\}$*, then apart from* p^* *there are three additional equilibria, namely,* $p^{*\prime} = (1, 1)$*,* $p^{*\prime\prime} = (51, 51)$ *and* $p^{*\prime\prime\prime} = (76, 76)$*. Notice that, although* A *is restrictive, it still leaves the bidders a fairly rich set of options.*

Several comments are in order. We have seen that in the first-price sealed-bid auction mechanism under complete information there might be *multiple equilibria*. Example 3.7 has shown that whether a tie at $a \in A$ is an equilibrium depends on the local properties of the bidding space at a, namely on the size of δ_a.[13] Moreover, when $v_{n-1} = v_n$ there are in many situations at least two equilibria.

The fact that in low price equilibria the revenue might be very low implies that the *seller has an incentive to prefer a second-price auction.* This is so because if the seller implements a second-price sealed-bid auction the usual dominance argument applies and the seller can be certain to obtain a higher revenue.[14]

Low price equilibria may generate considerably lower revenues than high price equilibria. In this sense there is '*collusion*'. But, given that bidding strategies constitute an equilibrium, they are also 'self-enforcing'. This contrasts with the conventional wisdom that "unlike in a second-price auction, the cartel agreement in a first-price auction is *not* self-enforcing and, hence, is somewhat fragile" (Krishna (2002), pg. 160).

A last comment concerns *efficiency*. In many instances the equilibrium is inefficient

[13] Notice also that the definition of the equilibrium strategies implies that the undominated Nash equilibrium is not unique, as a bidder with a low valuation may submit any undominated bid. Moreover, when $w_{n-1} = w_n$ and $\Gamma = \alpha$ there are two high price equilibria.

[14] This is in line with the results in Milgrom and Weber (1982) but not with Landsberger et al. (2001).

because the object is not awarded to the bidder with the highest valuation with positive probability.[15] Note that the intuition for this inefficiency is different from Landsberger et al. (2001). These authors obtain a similar result in the first-price sealed-bid auction with two bidders when the ranking of valuations is common knowledge. In their framework the inefficiency results from the fact that the low valuation bidder always bids higher (than a high valuation bidder with the same value).

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[15] In case (2.1) of Theorem 3.4 the equilibrium is inefficient. In addition, the same is true when case (2.2) applies but $v_{n-1} < v_n$ holds or when in case and (3) there exists $B_i \in W$ with $v_i < v_n$.

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